

# THE CHARACTERIZATION OF A LINE ARRANGEMENT WHOSE FUNDAMENTAL GROUP OF THE COMPLEMENT IS A DIRECT SUM OF FREE GROUPS

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**ABSTRACT.** Kwai Man Fan proved that if the intersection lattice of a line arrangement does not contain a cycle, then the fundamental group of its complement is a direct sum of infinite and cyclic free groups. He also conjectured that the converse is true as well. The main purpose of this paper is to prove this conjecture.

## 1. INTRODUCTION

An arrangement of lines is a finite collection of  $\mathbb{C}$ -linear subspaces of dimension 1. For such an arrangement  $\Sigma \subseteq \mathbb{C}^2$ , there is a natural projective arrangement  $\Sigma^*$  of lines in  $\mathbb{CP}^2$  associated to it, obtained by adding to each line its corresponding point at infinity. The problem of finding connections between the topology of the differentiable structure of  $\mathbb{C}^2 - \Sigma$  and the combinatorial theory of  $\Sigma$  is one of the main problems in the theory of line arrangements (see for example [CoSu]). The main motivations for studying the topology of  $\mathbb{C}^2 - \Sigma$  are derived from the areas of hypergeometric functions, singularity theory and algebraic geometry. Note that if 3 points are on the same line we do not consider it as a cycle.

Given an arrangement  $\Sigma$ , we define the graph  $G(\Sigma)$  which lies on the arrangement. Its vertices are the multiple points (with multiplicity  $\geq 3$ ) and the edges are the segments between multiple points on lines which pass through more than one multiple point. If two lines occur to meet in a simple point we ignore it (i.e. we do not consider it as a vertex of the graph).

In 1994, Jiang and Yau [JY] defined the concept of a "nice" arrangement. For  $\Sigma$ , they define a graph  $G(V, E)$ : The vertices are the multiple points of  $\Sigma$ .  $u, v$  are connected if there exists  $l \in \Sigma$  such that:  $u, v \in l$ . For  $v \in V$  define a subgraph  $G_\Sigma(v)$ : The vertex set is  $v$  and all his neighbors from  $\Sigma$  (pay attention this definition differs from the definition of fan).

$\Sigma$  is *nice* if there is  $V' \subset V$  such that  $E_\Sigma(v) \cap E_\Sigma(u) = \emptyset$  for all  $u, v \in V'$ , and if we delete the vertex  $v$  and the edges of its subgraph  $G_\Sigma(v)$  from  $G$ , for all  $v \in V'$  we get a forest (i.e. a graph without cycles).

They have proven several properties of those arrangements:

- (a) If  $A_1, A_2$  are nice arrangements and their lattices are isomorphic, then their complements  $\mathbb{C}^2 - A_1$  and  $\mathbb{C}^2 - A_2$  are diffeomorphic. This property naturally implies that  $\pi_1(\mathbb{C}^2 - A_1) \cong \pi_1(\mathbb{C}^2 - A_2)$ .

- (b) As a consequence of (a), they showed that the presentation of the fundamental group of the complement can be written explicitly and depends only on the lattice of the line arrangement.

In 2005, Wang and Yau [WY] continued this direction and proved that the results of Jiang-Yau hold for a much larger family of arrangements, which they call *simple arrangements*.

Falk [Flk] shows several examples of line arrangements with the same homotopy types but with different lattices. Fan [Fa1] proved that up to 6 lines the fundamental group of a real line arrangement is determined by the lattice Garber, Teicher and Vishne proved it for a real line arrangement with up to 8 lines.

Let  $G, H$  be groups that their abelianizations are free abelian groups of finite rank. Choudary, Dimca and Papadima [CDP] defined a set  $\Phi$  of natural group isomorphisms  $\phi : G \rightarrow H$  (which they call *1-marking*). In the case of  $G = \pi_1(\mathbb{C}^2 - \Sigma)$  (where  $\Sigma$  is an affine arrangement),  $\phi$  takes the topological structure of  $\mathbb{C}^2 - \Sigma$  into consideration. They prove that if  $A, B$  are line arrangements, and  $B$  is "nice" (in the meaning of Jiang-Yau), then the lattices are isomorphic if and only if there is an isomorphism  $\phi \in \Phi$  where  $\phi : \pi_1(\mathbb{C}^2 - A) \xrightarrow{\sim} \pi_1(\mathbb{C}^2 - B)$ .

Fan [Fa2] showed that if the graph  $G(\Sigma)$  is a forest (i.e. a graph without cycles), then the fundamental group is a direct sum of free groups. He also conjectured that the converse of his theorem is true. In [Fa3] Fan proved that if the fundamental group of the complement is a direct sum of free groups, then the arrangement is composed of parallel lines.

In this paper, we prove his conjecture. Our theorem will state that if the fundamental group is isomorphic to a direct sum of free groups, then the graph has no cycles. We would like to emphasize that we make no restrictions on our isomorphisms.

The structure of the paper is as follows. In Section 2, we give basic definitions related to groups. In section 3 we give basic definitions related to line arrangements and the fundamental group of the complement of line arrangements. In Section 4, we define a function and a special set induced by it. Section 5 deals with some special properties of fundamental groups of the complement of line arrangements which are direct sum of free groups. In Section 6, we prove the main result of the paper.

## 2. DEFINITIONS AND NOTATIONS

This section presents the needed definitions for the paper.

**2.1. Lower Central Series.** We start by defining the *lower central series of a group*  $G$  which will be used throughout the paper.

### Definition 2.1. Commutator group and lower central series

Let  $G$  be a group. The *commutator group* of  $G$  is

$$G' = G_2 = [G, G] = \langle \{aba^{-1}b^{-1} | a, b \in G\} \rangle$$

The subgroup  $G'$  is normal in  $G$  with an abelian quotient. We can define the *lower central series of  $G$*  recursively:

$$\begin{aligned} G_1 &= G \\ G_2 &= [G, G] \\ G_3 &= [G, G_2] \\ &\vdots \\ G_n &= [G, G_{n-1}]. \end{aligned}$$

Since  $G_{n+1}$  contains the commutators of  $G_n$ , we have  $G_{n+1} \triangleleft G_n$  and the quotient  $G_n/G_{n+1}$  is abelian for all  $n \in \mathbb{N}$ .

To understand these groups, the following identities are needed.

**Proposition 2.2. Witt-Hall Identities** [MKS]

- (1)  $[a, b][b, a] = e$ ;
- (2)  $[a, bc] = [a, b][a, c][[c, a], b]$ ;
- (3)  $[ab, c] = [a, [b, c]][b, c][a, c]$ .

From the second and third identities we get

**Lemma 2.3.** Let  $G$  be a group and let  $\{x_1, \dots, x_k\}$  be the generators of  $G$ . Then:

$$G_2/G_3 = \langle [x_i, x_j] \mid i \neq j, 1 \leq i, j \leq k \rangle.$$

*Proof.* If  $[xy, z] \in G_2$ , then

$$[xy, z] = xyz(xy)^{-1}z^{-1} = xyzy^{-1}x^{-1}z^{-1} = x[y, z]x^{-1}[x, z].$$

Over  $G_3$ , we know that  $[x, [y, z]] = x[y, z]x^{-1}[y, z]^{-1} = e \Rightarrow [y, z] = x[y, z]x^{-1}$ . We get

$$[xy, z] = [y, z][x, z] \text{ over } G_3.$$

This result means that  $G_2/G_3$  is finitely generated by the set

$$\{[x_i, x_j] \mid i \neq j, 1 \leq i, j \leq k\}$$

, where  $x_1, \dots, x_k$  are the generators of  $G$ . □

**Proposition 2.4.** Let  $G = \langle x_1, \dots, x_k \mid R \rangle$ , where the relations  $R$  are commutator type relations (i.e. every relation  $r \in R$  can be written as  $r = [w_1, w_2]$ ,  $w_1, w_2 \in G$ ). Then

$$G_2/G_3 = \left\langle [x_i, x_j] \middle| \begin{array}{l} [x_k, x_l] = [x_l, x_k]^{-1} \\ \text{all generators commute,} \\ R^* \end{array} \right\rangle$$

where  $R^*$  is set of the relations  $R$  written by means of the generators of  $G_2$ , taken modulo  $G_3$ .

The next Definition and theorem will give us a better understanding of  $G_2/G_3$  and help us in the future.

**Definition 2.5** ([Ha], p.165). Let  $G$  be a group generated by  $x_1, \dots, x_r$ . We consider formal words or strings  $b_1 \cdot b_2 \cdots b_n$  where each  $b$  is one of the generators. We also introduce formal commutators  $c_j$  and weights  $\omega(c_j)$  by the rules:

- (1)  $c_i = x_i, i = 1, \dots, r$  are the commutators of weight 1; i.e.  $\omega(x_i) = 1$ .
- (2) If  $c_i$  and  $c_j$  are commutators, then  $c_k = [c_i, c_j]$  and  $\omega(c_k) = \omega(c_i) + \omega(c_j)$ .

**Theorem 2.6. Basis Theorem** [Ha] *If  $F$  is the free group with free generators  $y_1, \dots, y_r$  and if in a sequence of basic commutators  $c_1, \dots, c_t$  are those of weights  $1, 2, \dots, n$  then an arbitrary element  $f \in F$  has a unique representation  $f = c_1^{e_1} c_2^{e_2} \cdots c_t^{e_t} \pmod{F_{n+1}}$ .*

*The basic commutators of weight  $n$  form a basis for the free abelian group  $F_n/F_{n+1}$ .*

## 2.2. Line Arrangements.

**Definition 2.7. Line arrangement.**

A line arrangement  $\Sigma = \{L_1, \dots, L_s\} \subseteq \mathbb{C}^2$  is a union of copies of  $\mathbb{C}^1$ .

**Remark 2.8.** Each time we are mentioning a line arrangement  $\Sigma \subseteq \mathbb{C}^2$ , we assume that there are no parallel lines in  $\Sigma$ . This assumption does not restrict us, since we can project every line arrangement in  $\mathbb{CP}^2$  to  $\mathbb{C}^2$  such that the new arrangement does not contain parallel lines.

**Definition 2.9.** Let  $\Sigma$  be a line arrangement. An intersection point in  $\Sigma$  is called *simple* if there are precisely two lines which meet at that point. Otherwise, we call it *multiple*.

**Definition 2.10. A Cycle**

A cycle is a non empty ordered set of multiple intersection points  $\{p_1, \dots, p_k\}$ , such that any pair of adjacent points  $p_j, p_{j+1}$  and the points  $p_1, p_k$  are connected by lines of the arrangement. Moreover, if  $1 \leq j \leq k-2$ , then the line connecting  $p_j$  to  $p_{j+1}$  and the line connecting  $p_{j+1}$  to  $p_{j+2}$  are different. Also, the line connecting  $p_{k-1}$  to  $p_k$  is different from the line connecting  $p_k$  to  $p_1$ .

Let  $\Sigma \subseteq \mathbb{CP}^2$  be a line arrangement. The invariant  $\beta(\Sigma)$  which is defined in [Fa2], counts the number of independent cycles in the graph.

Fan [Fa2] showed that  $\beta(\Sigma) = 0$  if and only if  $G(\Sigma)$  has no cycles.

In an analogous way for  $\Sigma \subseteq \mathbb{C}^2$ , we define  $\beta(\Sigma) := \beta(\Sigma \cup L_\infty)$  where  $L_\infty$  is the projective line at infinity of  $\mathbb{C}^2$  and  $G(\Sigma) := G(\Sigma \cup L_\infty)$ . Note that the new definitions are well defined.

One of the most important invariants of a line arrangement is the fundamental group of its complement, denoted by  $\pi_1(\mathbb{C}^2 - \Sigma)$ .

The next lemma presents its computation.

**Lemma 2.11. Constructing the Fundamental Group** ([Arv],[OT] [CoSu, p.304]) *Let  $\Sigma = \{L_1, \dots, L_n\} \subseteq \mathbb{C}^2$  be a line arrangement, that is enumerated as in [CoSu].*

*We associate a generator  $\Gamma_i$  to each line  $L_i$  such that*

$$G = \pi_1(\mathbb{C}^2 - \Sigma) = \langle \Gamma_1, \dots, \Gamma_n | R \rangle,$$

*where  $R$  is a set of relations generated as follows.*

Every intersection point of lines  $L_{i_1}, \dots, L_{i_m}$  creates a set of relations

$$\Gamma_{i_1}^{x_1} \Gamma_{i_2}^{x_2} \cdots \Gamma_{i_m}^{x_m} = \Gamma_{i_m}^{x_m} \Gamma_{i_1}^{x_1} \cdots \Gamma_{i_{m-1}}^{x_{m-1}} = \Gamma_{i_2}^{x_2} \cdots \Gamma_{i_m}^{x_m} \Gamma_{i_1}^{x_1}$$

where  $x_i \in G$  and  $\Gamma_i^{x_i} = x_i^{-1} \Gamma_i x_i$ .

It is easy to see that this set is equivalent to the following set:

$$[\Gamma_{i_j}^{x_j}, \Gamma_{i_1}^{x_1} \cdots \Gamma_{i_m}^{x_m}] = e, 1 \leq j \leq m.$$

We get a similar presentation for a real arrangement by using the Moishezon-Teicher algorithm [MoTe] and the van-Kampen Theorem [VK].

**Notation 2.12.** We denote by  $\mathcal{P}$  the set of intersection points. For  $p \in \mathcal{P}$ , we denote by  $\Gamma(p)$  the set of generators attached to the lines passing through the point  $p$  and by  $\Gamma(p)^c$  the set of generators attached to the lines not passing through the point  $p$ .

### 3. DECOMPOSITION OF $G_2/G_3$

Let  $G$  be a group. The abelianization of  $G$ , denoted by  $\overline{G}$ , is  $\overline{G} = G/G_2$ . If  $g \in G$ , we denote  $\overline{g} = g \cdot G_2$ .

**Remark 3.1.** Let  $\Sigma \in \mathbb{C}^2$  be a line arrangement, and let  $p \in \mathcal{P}$ . Then,  $G = \pi_1(\mathbb{C}^2 - \Sigma) = \langle \Gamma(p), \Gamma(p)^c | R \rangle$ ,

$$G/G_2 = \overline{G} = Ab(G) = \langle \overline{\Gamma(p)}, \overline{\Gamma(p)^c} | R, [x, y] = e, x, y \in \Gamma(p) \cup \Gamma(p)^c \rangle.$$

This Lemma is an immediate implementation of the last section.

**Lemma 3.2. An implementation for line arrangements.** Let  $\Sigma$  be a line arrangement, then the abelian group  $G_2/G_3$  can be written as

$$G_2/G_3 = \left\langle [\Gamma_i, \Gamma_j] \middle| \begin{array}{l} [\Gamma_i, \Gamma_j] = [\Gamma_j, \Gamma_i]^{-1}, \\ [\Gamma_i, \Gamma_j][\Gamma_k, \Gamma_l] = [\Gamma_k, \Gamma_l][\Gamma_i, \Gamma_j], \\ \prod_{\Gamma_x \in \Gamma(p)} [\Gamma_x, \Gamma_y], p \in \mathcal{P}, \Gamma_y \in \Gamma(p) \end{array} \right\rangle.$$

*Proof.* A simple implementation of Proposition 2.4 on the presentation of  $G$  from Lemma 2.11.  $\square$

**Remark 3.3.** We can see that if  $\Gamma_1$  and  $\Gamma_2$  are associated with lines meeting in one point and  $\Gamma_3$  and  $\Gamma_4$  are associated with lines meeting in a different point, there is no relation combining  $[\Gamma_1, \Gamma_2]$  and  $[\Gamma_3, \Gamma_4]$ . Therefore,

$$G_2/G_3 = \bigoplus_{p \in \mathcal{P}} C_p$$

$$\text{where } C_p = \left\langle [\Gamma_i, \Gamma_j], \Gamma_i, \Gamma_j \in \Gamma(p) \middle| \begin{array}{l} [\Gamma_i, \Gamma_j] = [\Gamma_j, \Gamma_i]^{-1} \\ [\Gamma_i, \Gamma_j][\Gamma_k, \Gamma_l] = [\Gamma_k, \Gamma_l][\Gamma_i, \Gamma_j], \Gamma_i, \Gamma_j, \Gamma_k, \Gamma_l \in \Gamma(p) \\ \prod_{\Gamma_x \in \Gamma(p)} [\Gamma_x, \Gamma_y], \Gamma_y \in \Gamma(p) \end{array} \right\rangle.$$

We can see that the generators of the different groups  $C_p$  in the direct sum are disjoint. Consequently, let  $x \in G_2/G_3$ , then  $x = \bigoplus_{p \in \mathcal{P}} c_p$ , where  $c_p \in C_p$ .

For each  $r \in \mathcal{P}$ , consider the projection

$$\xi_r : G_2/G_3 \rightarrow C_r$$

given by

$$\xi_r(x) = \xi_r\left(\bigoplus_{p \in \mathcal{P}} c_p\right) = c_r.$$

If  $\Gamma_i \in \Gamma(r)^c$ , then  $\forall \Gamma_j, \xi_r([\Gamma_i, \Gamma_j]) = \xi_r([\Gamma_j, \Gamma_i]) = e$ . If  $\Gamma_i, \Gamma_j \in \Gamma(r)$ , then  $\xi_r([\Gamma_i, \Gamma_j]) = [\Gamma_i, \Gamma_j]$ .

#### 4. THE STABILIZER OF AN INTERSECTION POINT

Let  $G = \pi_1(\mathbb{C}^2 - \Sigma)$ . Define

$$f : G/G_2 \times G/G_2 \longrightarrow G_2/G_3$$

by

$$f(\bar{a}, \bar{b}) = [a, b]/G_3.$$

This function is well-defined: If  $\bar{a}_1 = \bar{a}_2$  and  $\bar{b}_1 = \bar{b}_2$ , then  $a_2 = a_1x$  where  $x \in G_2$ ,  $b_2 = b_1y$  where  $y \in G_2$ . Then, by Proposition 2.2:

$$\begin{aligned} [a_2, b_2] &= [a_1x, b_1y] = [a_1x, b_1][a_1x, y] = [a_1x, b_1] = [a_1, b_1][x, b_1] = [a_1, b_1], \\ \text{so } f(\bar{a}_1, \bar{b}_1) &= f(\bar{a}_2, \bar{b}_2). \end{aligned}$$

The following lemma presents some properties of  $f$ :

**Lemma 4.1.** *Let  $a, b, c \in G/G_2$ . Then:*

- (1)  $f(a \cdot b, c) = f(a, c) \cdot f(b, c)$ .
- (2)  $f(a, b \cdot c) = f(a, b) \cdot f(a, c)$ .
- (3)  $f(a^n, b^m) = f(a, b)^{nm}$  for  $m, n \in \mathbb{Z}$ .
- (4)  $f(b, a) = (f(a, b))^{-1}$ .

*Proof.*

- (1) Let  $A, B, C \in G$  such that  $\bar{A} = a$ ,  $\bar{B} = b$ , and  $\bar{C} = c$ . This means that  $\bar{AB} = ab$ , so by definition,  $f(ab, c) = [AB, C]/G_3 = ([A, C][B, C])/G_3$  which by definition is equal to  $f(a, c)f(b, c)$ .
- (2) The proof is the same as (1).
- (3) Simple induction on (1) and (2).
- (4) Let  $A, B \in G$  such that  $\bar{A} = a$ ,  $\bar{B} = b$ . By definition

$$f(b, a) = [B, A]/G_3 = ([A, B]/G_3)^{-1} = (f(a, b))^{-1}$$

□

Now for any  $\bar{x} \in G/G_2$  we define

$$S(\bar{x}) = \{y \in G/G_2 | f(\bar{y}, \bar{x}) = e_{G_3}\}.$$

By Lemma 4.1,  $S(\bar{x})$  is a subgroup of  $G/G_2$ . From now on, we talk about a specific

intersection point  $Q$ . The lines passing through this point are  $\{L_{i_1}, \dots, L_{i_m}\}$ . Define:  $M = \Gamma_{i_1}^{x_1} \Gamma_{i_2}^{x_2} \cdots \Gamma_{i_m}^{x_m}$ . Then, as noted in Lemma 2.11, the relations induced from the point  $Q$  can be translated to  $[\Gamma_{i_1}^{x_1}, M] = \cdots = [\Gamma_{i_m}^{x_m}, M] = e$ .

Let  $\overline{M} = \overline{\Gamma_{i_1}} \cdot \overline{\Gamma_{i_2}} \cdots \overline{\Gamma_{i_m}}$ .

Since for each  $j$ ,  $\overline{\Gamma_{i_j}} = \overline{\Gamma_{i_j}^{x_j}}$  we get that

$$f(\overline{\Gamma_{i_j}}, \overline{M}) = [\Gamma_{i_j}^{x_j}, M]/G_3 = e_{G_3}. \quad (1)$$

**Theorem 4.2.** Let  $Q \in \mathcal{P}$  be an intersection point. Let  $\Gamma(Q) = \{\Gamma_{i_1}, \dots, \Gamma_{i_m}\}$  and  $M = \Gamma_{i_1}^{x_1} \cdots \Gamma_{i_m}^{x_m}$ .

Then

$$S(\overline{M}) = \left\langle \overline{\Gamma(Q)} \cup \left( \bigcap_{\Gamma \in \Gamma(Q)} S(\overline{\Gamma}) \right) \right\rangle.$$

We call  $S(\overline{M})$  the stabilizer of the intersection point  $Q$ .

*Proof.* We start by proving that  $S(\overline{M}) \supseteq \left\langle \overline{\Gamma(Q)} \cup \left( \bigcap_{\Gamma \in \Gamma(Q)} S(\overline{\Gamma}) \right) \right\rangle$ .

$\overline{\Gamma(Q)} \subseteq S(\overline{M})$ : We have already shown that if  $\Gamma \in \Gamma(Q)$ , then  $f(\overline{\Gamma}, \overline{M}) = \overline{e}$  and hence  $f(\overline{M}, \overline{\Gamma}) = f((\overline{\Gamma}, \overline{M}))^{-1} = \overline{e}$ .

$\bigcap_{\Gamma \in \Gamma(Q)} S(\overline{\Gamma}) \subseteq S(\overline{M})$ : If  $x \in \bigcap_{\Gamma \in \Gamma(Q)} S(\overline{\Gamma})$ , then for any  $\Gamma \in \Gamma(Q)$ ,  $f(\overline{\Gamma}, x) = e$ .

This means that also the product  $\prod_{\Gamma \in \Gamma(Q)} f(\overline{\Gamma}, x) = \overline{e}$ , so by Lemma 4.1

$$\overline{e} = \prod_{\Gamma \in \Gamma(Q)} f(\overline{\Gamma}, x) = f \left( \left( \prod_{\Gamma \in \Gamma(Q)} \overline{\Gamma} \right), x \right) = f(\overline{M}, x).$$

Therefore  $x \in S(\overline{M})$ .

Since  $S(\overline{M})$  is a subgroup and we have shown that it contains the union of  $\overline{\Gamma(Q)}$  and  $\bigcap_{\Gamma \in \Gamma(Q)} S(\overline{\Gamma})$ , it clearly contains the subgroup generated by the union.

To complete the proof, we prove the opposite inclusion:

$$S(\overline{M}) \subseteq \left\langle \overline{\Gamma(Q)} \cup \left( \bigcap_{\Gamma \in \Gamma(Q)} S(\overline{\Gamma}) \right) \right\rangle.$$

Let  $\overline{x} \in S(\overline{M}) \subseteq G/G_2 = \langle \overline{\Gamma(Q)}, \overline{\Gamma(Q)^c} | \overline{R} \rangle$ . Then  $\overline{x}$  can be written as  $\overline{x} = \overline{z} \cdot \overline{y}$  where  $\overline{z} \in \langle \overline{\Gamma(Q)} \rangle$ ,  $\overline{y} \in \langle \overline{\Gamma(Q)^c} \rangle$ . We will prove  $\overline{y} \in \bigcap_{\Gamma \in \Gamma(Q)} S(\overline{\Gamma})$ .

Since  $\overline{z^{-1}} \in \langle \overline{\Gamma(Q)} \rangle \subseteq S(\overline{M})$ , so  $\overline{y} = \overline{z^{-1}} \cdot \overline{z} \cdot \overline{y} = \overline{z^{-1}x}$ . Hence we get  $\overline{y} \in S(\overline{M})$ .

Let  $l_x$  be a line passing through  $Q$  and  $\Gamma_x$  be the generator associated with it. Recall that  $\overline{y} \in S(\overline{\Gamma})$  if  $f(\overline{\Gamma}, \overline{y}) = \overline{e}$ . We need to prove that  $f(\overline{\Gamma_x}, \overline{y}) = \overline{e}$  in  $G_2/G_3$ . From Lemma 3.3 we know that  $G_2/G_3$  is a direct sum of groups

$$G_2/G_3 = \bigoplus_{p \in \mathcal{P}} C_p,$$

so it remains to prove that  $f([\overline{\Gamma_x}, \overline{y}]) = \overline{e}$  in  $C_p$ , for all  $p \in \mathcal{P}$ . Let  $p \in \mathcal{P}$ . We have to show that the projection of the coset  $[\overline{\Gamma_x}, \overline{y}]/G_3$  on  $C_p$  is trivial.

We separate our proof into three cases.

**Case 1:**  $p = Q$ .

Since  $\overline{y} \in \langle \overline{\Gamma(Q)^c} \rangle = \langle \overline{\Gamma(p)^c} \rangle$  then the projection is trivial by the definition of  $C_p$ .

**Case 2:**  $l_x$  does not pass through  $p$ .

In this case,  $\overline{\Gamma_x} \in \langle \overline{\Gamma(p)^c} \rangle$  and therefore the projection is trivial, by the definition of  $C_p$ .

**Case 3:**  $p \neq Q$  and  $l_x$  does pass through  $p$ .

By definition of  $S$ ,  $f(\prod_{\Gamma \in \Gamma(Q)} \overline{\Gamma}, \overline{y}) = \overline{e}$ . This means by the properties of  $f$  (Lemma 4.1) that  $\prod_{\Gamma \in \Gamma(Q)} f(\overline{\Gamma}, \overline{y}) = \overline{e}$ . Since  $l_x$  passes through  $p$ , all the other lines passing through  $Q$  do not cross  $p$ . So if we project any  $f(\overline{\Gamma_z}, \overline{y})$  where  $\Gamma_z$  is any other generator from  $\Gamma(Q)$ , we get the identity. So in  $C_p$ ,

$$\overline{e} = \prod_{\Gamma \in \Gamma(Q)} f(\overline{\Gamma}, \overline{y}) = f(\overline{\Gamma_x}, \overline{y}),$$

and thus  $f(\overline{\Gamma_x}, \overline{y}) = \overline{e}$ .

□

## 5. FUNDAMENTAL GROUPS WHICH ARE SEMIDIRECT PRODUCT

**Theorem 5.1.** *Let  $G = (\bigoplus_{i=1}^n A_i) \oplus \mathbb{Z}^l$  where  $A_i$  are free groups and let  $\psi : G \rightarrow G$  be a projection where  $\text{im}(\psi) = \langle y_1, \dots, y_k \rangle \cong \mathbb{F}_k$ . Then, there exists  $r, 1 \leq r \leq n$ , such that  $\forall g \in G, \text{pr}_r(g) = e$  implies that  $\psi(g) = e$  and  $\psi = \psi \circ \text{pr}_r$ , where  $\text{pr}_r : G \rightarrow G$  is the projection onto the subgroup of  $G$  naturally isomorphic to  $A_r$ .*

*Proof.* Let us denote the  $\mathbb{Z}^l$  component as  $A_0$ . Since  $\psi$  is a projection,  $\psi(y_1) = y_1$  and  $\psi(y_2) = y_2$ . Then  $\psi([y_1, y_2]) = [y_1, y_2] \neq e$ . Note:

$$[y_1, y_2] = \bigoplus_{i=1}^n \text{pr}_i([y_1, y_2]) + \text{pr}_0([y_1, y_2]).$$

Since  $A_0$  is abelian,  $\text{pr}_0([y_1, y_2]) = e$ , and therefore

$$[y_1, y_2] = \bigoplus_{i=1}^n \text{pr}_i([y_1, y_2]).$$

Since  $\psi$  is a homomorphism:

$$\psi([y_1, y_2]) = \psi(\text{pr}_1([y_1, y_2]) \oplus \dots \oplus \text{pr}_n([y_1, y_2])) = \psi(\text{pr}_1([y_1, y_2])) \oplus \dots \oplus \psi(\text{pr}_n([y_1, y_2])) \neq e.$$

This means that  $\exists r, 1 \leq r \leq n$ , such that  $\psi(\text{pr}_r[y_1, y_2]) \neq e$ , so if we denote  $a := \text{pr}_r(y_1)$  and  $b := \text{pr}_r(y_2)$  we get  $[\psi(a), \psi(b)] \neq e$ .

Since  $a, b \in A_r$ , if  $x \in A_j$  where  $j \neq r$ , then  $[x, a] = [x, b] = e$ . We get that

$$[\psi(x), \psi(a)] = [\psi(x), \psi(b)] = e.$$

So  $\psi(x)$  commutes with two noncommutative elements in a free group and hence:  $\psi(x) = e, \forall x \in A_j, j \neq r$ . This means that if  $g \in G$  and  $\text{pr}_r(g) = e$ , then  $\psi(g) = e$ .

For each  $g \in G$ ,  $g = v_1 \oplus \cdots \oplus v_n \oplus w$ ,  $v_j \in A_j$ ,  $w \in \mathbb{Z}^l$ , we have that  $\psi(g) = \psi(v_1 \oplus \cdots \oplus v_n \oplus w) = \psi(v_1) \oplus \cdots \oplus \psi(v_n) \oplus \psi(w) = \psi(v_r) = \psi(\text{pr}_r(g))$ . Therefore  $\psi = \psi \circ \text{pr}_r$ .

□

For proving the next result, we use the following theorem from [LS]:

**Theorem 5.2.** *Let  $\phi$  be an epimorphism from a finitely generated free group  $F$  to a free group  $G$ . Then  $F$  has a basis  $Z = Z_1 \cup Z_2$  such that  $\phi$  maps  $\langle Z_1 \rangle$  isomorphically onto  $G$  and maps  $\langle Z_2 \rangle$  to  $e$ .*

**Corollary 5.3.** With the same assumptions of Theorem 5.1 where  $\text{im}(\psi) = \langle y_1, \dots, y_k \rangle \cong \mathbb{F}_k$ , for the same  $r$  in Theorem 5.1 we can find elements  $\{z_1, \dots, z_m | m \geq 0\} \subseteq A_r$  such that

$$\{\text{pr}_r(y_1), \dots, \text{pr}_r(y_k), z_1, \dots, z_m | m \geq 0\}$$

are generators of  $A_r$  and  $\psi(z_i) = e$ .

*Proof.* From Theorem 5.1 and the fact that  $\psi$  is projection ( $\psi \circ \psi = \psi$ ), we get that  $\text{pr}_r \circ \psi \circ \text{pr}_r \circ \psi = \text{pr}_r \circ \psi$ . Therefore,  $\text{pr}_r \circ \psi$  is a projection.

Now since  $(\text{pr}_r \circ \psi)(A_r) \subseteq A_r$ ,

$$(\text{pr}_r \circ \psi)|_{A_r} \circ (\text{pr}_r \circ \psi)|_{A_r} = ((\text{pr}_r \circ \psi) \circ (\text{pr}_r \circ \psi))|_{A_r} = (\text{pr}_r \circ \psi)|_{A_r}.$$

Given:

$$\text{im}(\psi) = \langle y_1, \dots, y_k \rangle \text{ is a free group.} \quad (*)$$

Now we claim that:  $\{\text{pr}_r(y_1), \dots, \text{pr}_r(y_k)\}$  are the generators of free group  $\mathbb{F}_k$ . Otherwise, there exists a nontrivial word  $w = W(y_1, \dots, y_k)$  so that  $\text{pr}_r(w) = W(\text{pr}_r(y_1), \dots, \text{pr}_r(y_k)) = e$ . Therefore:

$$(\psi \circ \text{pr}_r)(w) = \psi(\text{pr}_r(w)) = \psi(e) = e.$$

By Theorem 5.1:  $\psi = \psi \circ \text{pr}_r$ , therefore  $\psi(w) = (\psi \circ \text{pr}_r)(w) = e$  which means that  $w = e$ , a contradiction to (\*).

In conclusion, we get that  $(\text{pr}_r \circ \psi)|_{A_r} : A_r \rightarrow A_r$  is a projection, such that  $\text{im}(\text{pr}_r \circ \psi) = \langle \text{pr}_r(y_1), \dots, \text{pr}_r(y_k) \rangle \cong \mathbb{F}_k$  ( $\text{pr}_r(y_1), \dots, \text{pr}_r(y_k)$  are the generators of  $\mathbb{F}_k$ ).

By Theorem 5.2,  $A_i$  has a basis  $B = B_1 \cup B_2$  such that  $\text{im}(\text{pr}_r \circ \psi) = \langle B_1 \rangle$  and  $(\text{pr}_r \circ \psi)(B_2) = e$ . Since  $\text{pr}_r \circ \psi$  is a projection, we can assume that  $B_1 = \langle \text{pr}_r(y_1), \dots, \text{pr}_r(y_k) \rangle$ . Let  $B_2 = \{z_1, \dots, z_m | \text{pr}_r \circ \psi(z_i) = e\}$ , which means that  $(\psi \circ \text{pr}_r \circ \psi)(z_i) = e$ . As we proved earlier,  $\psi \circ \text{pr}_r = \psi$ , so  $(\psi \circ \psi)(z_i) = e$ . But  $\psi \circ \psi = \psi$ , therefore  $\psi(z_i) = e$ .

This implies that  $A_r$  has generators

$$\{\text{pr}_r(y_1), \dots, \text{pr}_r(y_k), z_1, \dots, z_m\}$$

which satisfy the needed properties.

□

Without loss of generality, we temporarily change to a local numeration.

Let  $G = \pi_1(\mathbb{C}^2 - \Sigma) = \langle \Gamma_1, \dots, \Gamma_n, \Gamma_{n+1}, \dots, \Gamma_l | R \rangle$ , where  $\Gamma(A) = \{\Gamma_1, \dots, \Gamma_n\}$  are the generators of the lines which participate in a specific multiple intersection point called A.

**Definition 5.4.** [Jo] Let  $G$  be a group with a given presentation:  $G = \langle X | R \rangle$ . A *Tietze transformation*  $T_i$  ( $1 \leq i \leq 4$ ) is a transformation of  $\langle X | R \rangle$  to a presentation  $G = \langle X' | R' \rangle$  of one of the following types:

- (T<sub>1</sub>) If  $r$  is a word in  $\langle X \rangle$  and  $r = e$  is a relation that holds in  $G$ , then let  $X' = X, R' = R \cup \{r\}$ .
- (T<sub>2</sub>) If  $r \in R$  is such that the relation  $r = e$  holds in the group  $\langle X | R - r \rangle$ , then let  $X' = X, R' = R \setminus \{r\}$ .
- (T<sub>3</sub>) If  $w$  is a word in  $\langle X \rangle$  and  $z \notin X$ , put  $X' = X \cup \{z\}, R' = R \cup \{wz^{-1}\}$ .
- (T<sub>4</sub>) If  $z \in X$  and  $w$  is a word in the other elements of  $X$  such that  $wz^{-1} \in R$ , then substitute  $w$  for  $z$  in every other element of  $R$  to get  $\tilde{R}$  and take  $X' = X - \{z\}, R' = \tilde{R}$ .

Before introducing the notion of a *braid monodromy* [MoTe], we have to make some constructions. From now, we will work in  $\mathbb{C}^2$ . Let  $E$  (resp.  $D$ ) be a closed disk on  $x$ -axis (resp.  $y$ -axis), and let  $C$  be a part of an algebraic curve in  $\mathbb{C}^2$  located in  $E \times D$ . Let  $\pi_1 : E \times D \rightarrow E$  and  $\pi_2 : E \times D \rightarrow D$  be the canonical projections, and let  $\pi = \pi_1|_C : C \rightarrow E$ . Assume  $\pi$  is a proper map, and  $\deg \pi = n$ . Let  $N = \{x \in E \mid \#\pi^{-1}(x) < n\}$ , and assume  $N \cap \partial E = \emptyset$ . Now choose  $M \in \partial E$  and let  $K = K(M) = \pi^{-1}(M)$ . By the assumption that  $\deg \pi = n$  ( $\Rightarrow \#K = n$ ), we can write:  $K = \{a_1, a_2, \dots, a_n\}$ . Under these constructions, from each loop in  $E - N$ , we can define a braid in  $B_n[M \times D, K]$  in the following way:

- (1) Since  $\deg \pi = n$ , we can lift any loop in  $E - N$  with a base point  $M$  to a system of  $n$  paths in  $(E - N) \times D$  which start and finish at  $\{a_1, a_2, \dots, a_n\}$ .
- (2) Project this system into  $D$  (by  $\pi_2$ ), to get  $n$  paths in  $D$  which start and end at the image of  $K$  in  $D$  (under  $\pi_2$ ). These paths actually form a motion.
- (3) Induce a braid from this motion, see [Ga].

To conclude, we can match a braid to each loop. Therefore, we get a map  $\alpha : \pi_1(E - N, M) \rightarrow B_n[M \times D, K]$ , which is also a group homomorphism. This map is called the **braid monodromy of  $C$  with respect to  $E \times D, \pi_1, M$**  (see [MoTe], [Ga]).

The following remark demonstrates the necessity of the condition that there are no parallel lines in the affine plane :

**Remark 5.5.** Let  $\Sigma$  be a line arrangement with no parallel line in the affine plane. Let  $\Gamma_1, \dots, \Gamma_l$  be the generators of  $\pi_1(\mathbb{C}^2 - \Sigma)$ . Then

$$\Gamma_1 \cdots \Gamma_l \in Z(\pi_1(\mathbb{C}^2 - \Sigma)).$$

*Proof.* Let  $\alpha : \pi_1(\mathbb{C} - \Sigma) \rightarrow B_n$  be the braid monodromy of  $\Sigma$ . By Remark 4.7 in [CoSu], the closed braid determined by the product  $\alpha(\Gamma_1) \cdots \alpha(\Gamma_l)$  is actually a link of the curve at infinity. In a generic curve, we have:  $\alpha(\Gamma_1) \cdots \alpha(\Gamma_l) = \Delta^2$ , where  $\Delta$  is the braid which rotates by  $180^\circ$  counterclockwise all the strands together. One of the presentations of the fundamental group is:  $\langle \Gamma_1, \dots, \Gamma_l | \alpha(s_i)(\Gamma_j) = \Gamma_j \forall i, \forall j \rangle$ , where  $s_i$  is the loop created in the  $x$ -axis as a result of the projection of the line arrangement on the plane [VK]. Therefore,  $\alpha(s_i)$  is a braid acting on the generators of the fundamental group. As a result  $\alpha(s_1 \cdots s_n)(\Gamma_j) = \Gamma_j, \forall j$ , which

means  $\Delta^2(\Gamma_j) = \Gamma_j, \forall j$ . It is known that  $\Delta^2(x) = x^{\Gamma_1 \cdots \Gamma_l}$  and thus  $\Gamma_j^{\Gamma_1 \cdots \Gamma_l} = \Gamma_j$ . Hence,  $\Gamma_1 \cdots \Gamma_l \in Z(\pi_1(\mathbb{C}^2 - \Sigma))$ .  $\square$

Let  $G = \pi_1(\mathbb{C}^2 - \Sigma) = \langle \Gamma_1, \dots, \Gamma_n, \Gamma_{n+1}, \dots, \Gamma_l | R \rangle$ , where  $\Gamma(A) = \{\Gamma_1, \dots, \Gamma_n\}$  are the generators of the lines which pass through in a specific multiple intersection point called A.

Let us recall that the lines passing through  $Q$  are  $\{L_{i_1}, \dots, L_{i_m}\}$ . Recall also  $M = \Gamma_{i_1}^{x_1} \Gamma_{i_2}^{x_2} \cdots \Gamma_{i_m}^{x_m}$ .

**Theorem 5.6.** *With the same assumptions of Theorem 5.1 and Remark 5.5, let  $H := \langle \Gamma_{i_1}, \dots, \Gamma_{i_{m-1}} \rangle$ . Let  $N$  be the normal closure of  $\{\Gamma(Q)^c, M\}$ . Let  $f_1 : H \rightarrow G$  be the natural embedding and  $f_2 : G \rightarrow G/N$  the natural homomorphism. Then:*

- (1)  $G/N \cong \mathbb{F}_{m-1}$  which is generated by  $f_2(\Gamma_{i_j})$ ,  $1 \leq j \leq m-1$ ;
- (2)  $H \cong \mathbb{F}_{m-1}$ ;
- (3)  $G = N \rtimes \mathbb{F}_{m-1}$ ;
- (4) There exists a projection  $h : G \rightarrow G$  ( $h^2 = h$ ), such that  $\text{im}(h) = H$  and  $\ker(h) = N$ .

*Proof.* (1) We can present  $G/N = \langle \Gamma(Q), \Gamma(Q)^c | R, M, \Gamma(Q)^c \rangle$ . Denote  $\Gamma(Q)^c = \{z_1, \dots, z_k\}$ . By applying iteratively Tietze's transformation ( $T_4$ ) for every  $z_i \in \Gamma(Q)^c$ , we obtain  $G/N = \langle \Gamma(Q) | \widehat{R}, \widehat{M} \rangle$ , where  $\widehat{R}, \widehat{M}$  are obtained from  $R, M$  by substituting  $e$  for  $z_i$ ,  $1 \leq i \leq k$ , in every other element of  $R$  and  $M$ , respectively.

Let  $v \in \mathcal{P}$  with  $v \neq Q$  be an intersection point, and let  $r$  be the relations induced by  $v$ .

Now we can get the following:

- (a) If there is a line  $L$  that goes through both points  $v$  and  $Q$  and the generator attached to  $L$  is  $\Gamma_{i_j}$  for some  $j$ ,  $1 \leq j \leq m$ , then the relations  $r$  become  $[\Gamma_{i_j}^x, 1, \dots, 1]$  for some  $x$ . These relations are trivial.
- (b) If the points  $v$  and  $Q$  do not share any line, then the relations  $r$  become  $[1, \dots, 1]$ , which are also trivial.

Let us denote by  $\widetilde{W}$  the word  $W$  after rewriting. Due to the above observations, using the presentation  $G/N = \langle \Gamma(Q) | \widehat{R}, \widehat{M} \rangle$ , then

$$G/N = \langle \Gamma_{i_1}, \dots, \Gamma_{i_m} | [\Gamma_{i_1}^{\widetilde{x_{i_1}}}, \dots, \Gamma_{i_m}^{\widetilde{x_{i_m}}}], \widetilde{M} \rangle.$$

This is equal to

$$\langle \Gamma_{i_1}, \dots, \Gamma_{i_m} | [\Gamma_{i_1}^{\widetilde{x_{i_1}}}, \widetilde{M}], \dots, [\Gamma_{i_m}^{\widetilde{x_{i_m}}}, \widetilde{M}], \widetilde{M} \rangle = \mathbb{F}_{m-1}.$$

- (2) Assume otherwise. Then there exists a non-trivial word  $w(\Gamma_{i_1}, \dots, \Gamma_{i_{m-1}}) = e$ . So applying  $f_2$ , we get a non-trivial word  $w(f_2(\Gamma_{i_1}), \dots, f_2(\Gamma_{i_{m-1}}))$  which is not possible by (1).
- (3) From the first paragraph, we get that  $f_2 \circ f_1$  is a surjective function from  $\mathbb{F}_{m-1}$  to  $\mathbb{F}_{m-1}$  and therefore it is an isomorphism.
- (4) Derived directly from (2) and (3).  $\square$

By the last theorem: Let  $Q \in \mathcal{P}$ ,  $\Gamma(Q) = \{\Gamma_{i_1}, \dots, \Gamma_{i_m}\}$ . Then there is a projection  $h : G \rightarrow G$  such that  $\text{im}(h) = \langle \Gamma_{i_1}, \dots, \Gamma_{i_m} \rangle$  and  $\ker(h)$  is the normal closure of  $\langle \Gamma(Q)^c, M \rangle$ .

By Theorem 5.1 and Corollary 5.3 we get that  $\exists r$ ,  $1 \leq r \leq n$ , such that  $A_r = \{w_1, \dots, w_{n-1}, z_1, \dots, z_k | k \geq 0\}$  and  $h = h \circ \text{pr}_r$ ,  $\text{pr}_r \circ h$  is a projection,  $\Gamma_i = h(w_i)$ ,  $h(z_i) = e$ ,  $b_i = \Gamma_i w_i^{-1}$ ,  $h(b_i) = e$  (note that  $h$  has the role of  $\psi$ ).

**Claim 5.7.**  $\overline{A_r} = \text{pr}_{\bar{r}}(S(\overline{M}))$ .

*Proof.* We start by proving that  $\text{pr}_r(M) = e$ : for all  $x, y$  and  $j_1 \neq j_2$ . Since  $\Gamma_{i_{j_1}}$  and  $\Gamma_{i_{j_2}}$  are different generators of the free group  $\text{im}(h)$ :  $h([\Gamma_{i_{j_1}}^x \Gamma_{i_{j_2}}^y]) \neq e$ . We know that  $h = h \circ \text{pr}_r$ , so we get  $(h \circ \text{pr}_r)([\Gamma_{i_{j_1}}^x \Gamma_{i_{j_2}}^y]) = h([\Gamma_{i_{j_1}}^x \Gamma_{i_{j_2}}^y]) \neq e$ , therefore  $\text{pr}_r([\Gamma_{i_{j_1}}^x \Gamma_{i_{j_2}}^y]) \neq e$ . Hence, for  $x = x_{j_1}, y = x_{j_2}$ ,  $\text{pr}_r([\Gamma_{i_{j_1}}^{x_{j_1}}, \Gamma_{i_{j_2}}^{x_{j_2}}]) \neq e$ . We know that  $[\Gamma_{i_{j_1}}^{x_{j_1}}, M] = e$ , so the elements  $\text{pr}_r(\Gamma_{i_j}^{x_j})$ ,  $j = 1, 2$ , commute with  $M$  but do not commute with each other in the free group  $A_r$ , therefore  $\text{pr}_r(M) = e$ .

The next step is to prove that  $\overline{A_r} \subseteq S(\overline{M})$ . Let  $a \in \overline{A_r}$ . Then there exists  $a^* \in A_r$  such that  $\overline{a^*} = a$ . Since  $\text{pr}_r(M) = e$  and  $a^* \in A_r$ , by the properties of direct sum  $[a^*, M] = e$ . Therefore  $[a^*, M]/G_3 = e_{G_3}$ . By definition  $f(a, \overline{M}) = e_{G_3}$  which implies that  $a \in S(\overline{M})$ .

Hence, we have  $\overline{A_r} \subseteq \text{pr}_{\bar{r}}(S(\overline{M}))$ . The opposite inclusion is trivial by definition, and thus  $\overline{A_r} = \text{pr}_{\bar{r}}(S(\overline{M}))$ .  $\square$

By Theorem 4.2,  $S(\overline{M}) = \langle \{\overline{\Gamma_{i_1}}, \dots, \overline{\Gamma_{i_m}}\} \cup \{\bigcap_{k=1}^m S(\overline{\Gamma_{i_k}})\} \rangle$ , therefore:

$$\begin{aligned} \overline{A_r} &= \text{pr}_{\bar{r}} \left( \left\langle \{\Gamma_{i_1}, \dots, \Gamma_{i_m}\} \cup \bigcap_{k=1}^m S(\Gamma_{i_k}) \right\rangle \right) \\ &= \left\langle \{\text{pr}_{\bar{r}}(\Gamma_{i_1}), \dots, \text{pr}_{\bar{r}}(\Gamma_{i_m})\} \cup \text{pr}_{\bar{r}} \left( \bigcap_{k=1}^m S(\overline{\Gamma_{i_k}}) \right) \right\rangle. \end{aligned}$$

We claim that the right part of this generating set is trivial:

**Claim 5.8.**

$$\text{pr}_{\bar{r}} \left( \bigcap_{k=1}^m S(\overline{\Gamma_{i_k}}) \right) = \{e\}.$$

*Proof.* It is known that for any two sets  $A, B$  and a function  $F$  that  $F(A \cap B) \subseteq F(A) \cap F(B)$ . Therefore  $\text{pr}_{\bar{r}} \left( \bigcap_{k=1}^m S(\overline{\Gamma_{i_k}}) \right) \subseteq \bigcap_{k=1}^m [\text{pr}_{\bar{r}}(S(\overline{\Gamma_{i_k}}))]$ .

As we mentioned  $A_r = \langle \text{pr}_r(\Gamma_1), \dots, \text{pr}_r(\Gamma_m), z_1, \dots, z_q | q \geq 0 \rangle$ .

In other words:  $A_r = \langle \beta_1, \dots, \beta_{m+q} \rangle$  where

$$\beta_k = \begin{cases} \text{pr}_r(\Gamma_{i_k}) & k \leq m \\ z_{k-m} & m+1 \leq k \leq m+q \end{cases}$$

Let  $\bar{g} \in \text{pr}_{\bar{r}}(S(\Gamma_{i_k})) \subseteq \overline{A_r}$ , therefore  $\bar{g} = \overline{\beta_1^{t_1}} \cdots \overline{\beta_{q+m}^{t_{m+q}}}$ . Without loss of generality, we can choose that  $g = \beta_1^{t_1} \cdots \beta_{m+q}^{t_{m+q}}$ . By Theorem 5.1 we know that

$$G = A_1 \oplus \cdots \oplus A_r \oplus \cdots \oplus A_n \oplus \mathbb{Z}^\ell.$$

We define  $D = A_1 \oplus \cdots \oplus \widehat{A_r} \oplus \cdots \oplus A_n \oplus \mathbb{Z}^\ell$ , then  $G = A_r \oplus D$ . Hence, there exists  $g_2 \in D$  such that  $\bar{g} \oplus \bar{g}_2 \in S(\overline{\Gamma_{i_k}})$ .

Let  $\varphi : G \rightarrow D$  be the natural projection, and define  $\eta_k := \varphi(\Gamma_{i_k})$ . Hence, it implies that  $\Gamma_{i_k} = \text{pr}_r(\Gamma_{i_k}) \oplus \eta_k$ .

$$\begin{aligned} e_{G_3} &= f(\overline{\Gamma_{i_k}}, \overline{g \oplus g_2}) = f(\overline{\text{pr}_r(\Gamma_{i_k}) \oplus \eta_k}, \overline{g \oplus g_2}) \\ &= f(\overline{\text{pr}_r(\Gamma_{i_k})}, \overline{g}) f(\overline{\text{pr}_r(\Gamma_{i_k})}, \overline{g_2}) f(\overline{\eta_k}, \overline{g}) f(\overline{\eta_k}, \overline{g_2}). \end{aligned}$$

By the definition of  $f$ , this implies that

$$[\overline{\text{pr}_r(\Gamma_{i_k})}, \overline{g}] [\overline{\text{pr}_r(\Gamma_{i_k})}, \overline{g_2}] [\overline{\eta_k}, \overline{g}] [\overline{\eta_k}, \overline{g_2}] \in G_3.$$

Since  $\text{pr}_r(\Gamma_{i_k}), g \in A_r$ , and  $g_2, \eta_k \in D$ , then  $[\text{pr}_r(\Gamma_{i_k}), g_2] = [\eta_k, g] = e_{G_3}$ . Therefore  $[\text{pr}_r(\Gamma_{i_k}), g][\eta_k, g_2] \in G_3 = (A_r)_3 \oplus D_3$ , which means  $[\text{pr}_r(\Gamma_{i_k}), g] \in (A_r)_3, [\eta_k, g_2] \in D_3$ .

The fact that  $(A_r)_2/(A_r)_3 = \langle [\beta_i, \beta_j] | 1 \leq i < j \leq m+q \rangle$  is derived directly from the definition of  $A_r$ . Hence,  $(A_r)_3 \ni [\text{pr}_r(\Gamma_{i_k}), g] = [\beta_k, g] = [\beta_k, \beta_1^{t_1} \cdots \beta_{m+q}^{t_{m+q}}] = [\beta_k, \beta_1]^{t_1} \cdots [\beta_k, \beta_k]^{t_k} \cdots [\beta_k, \beta_{m+q}]^{t_{m+q}}$ .

Note that  $[\beta_k, \beta_k] = e$ . By [Ha],  $\{[\beta_k, \beta_j] | j \neq k\}$  are generators (or inverses) of  $(A_r)_2/(A_r)_3$  which is a free abelian group. Since every generator appears at most once, we get  $t_j = 0$  for all  $j \neq k$ .

As a conclusion,  $\bar{g} = \overline{\beta_k^{t_k}} \in \langle \overline{\beta_k} \rangle = \langle \overline{\text{pr}_r(\Gamma_{i_k})} \rangle$ . So  $\bar{g} \in \langle \text{pr}_{\bar{r}}(\Gamma_{i_k}) \rangle$ . To summarize, we have shown that if  $\bar{g} \in \langle \text{pr}_{\bar{r}}(S(\Gamma_{i_k})) \rangle$  then  $\bar{g} \in \langle \text{pr}_{\bar{r}}(\Gamma_{i_k}) \rangle$ . Therefore,  $\langle \text{pr}_{\bar{r}}(S(\Gamma_{i_k})) \rangle \subseteq \langle \text{pr}_{\bar{r}}(\Gamma_{i_k}) \rangle$ . As a conclusion,  $\bigcap_{k=1}^m \langle \text{pr}_{\bar{r}}(S(\Gamma_{i_k})) \rangle \subseteq \bigcap_{k=1}^m \langle \text{pr}_{\bar{r}}(\Gamma_{i_k}) \rangle = e$ , and hence  $\text{pr}_{\bar{r}}\left(\bigcap_{k=1}^m S(\Gamma_{i_k})\right) = \{e\}$  as claimed.  $\square$

Since  $\Gamma_{i_1} \cdots \Gamma_{i_m} = e$ ,  $\overline{A_r} = \langle \overline{\Gamma_{i_1}}, \dots, \overline{\Gamma_{i_{m-1}}} \rangle$ . As a result,  $\overline{A_r} = \text{pr}_{\bar{r}}\langle \Gamma_{i_1}, \dots, \Gamma_{i_{m-1}} \rangle$ . Consequently,  $\text{rank}(A_r) \leq m-1$ .

We know that  $A_r = \langle w_1, \dots, w_{m-1}, z_1, \dots, z_t | t \geq 0 \rangle$ . Combining these two facts together and by Corollary 5.2, we get  $A_r = \langle w_1, \dots, w_{m-1} \rangle$ , where  $y_i = (\text{pr}_r \circ h)(\Gamma_{i_j}^{x_j})$ .

From Theorem 5.1,  $h \circ \text{pr}_r = h$ . Since  $h$  is a projection  $h^2 = h$ . Hence, we get :

$$\text{pr}_r \circ h = \text{pr}_r \circ h \circ \text{pr}_r \circ h = \text{pr}_r \circ h \circ h = \text{pr}_r \circ h,$$

which means that  $\text{pr}_r \circ h$  is a projection too. Recall that

$$A_r = \langle (\text{pr}_r \circ h)(\Gamma_{i_1}), \dots, (\text{pr}_r \circ h)(\Gamma_{i_{m-1}}) \rangle.$$

Therefore  $(\text{pr}_r \circ h)(w_j) = (\text{pr}_r \circ h) \circ (\text{pr}_r \circ h)(\Gamma_{i_j}) = (\text{pr}_r \circ h)(\Gamma_{i_j}) = w_j$ . Hence we get that  $(\text{pr}_r \circ h)(A_r) = A_r$ .

Now if  $l_i$  is a line that does not pass through the point  $Q$  and  $\Gamma_i$  is the generator of  $l_i$ , then  $h(\Gamma_i) = e$  and therefore  $(\text{pr}_r \circ h)(\Gamma_i) = \text{pr}_r(e) = e$ .

From this investigation, we get the following theorem.

**Theorem 5.9.** *Let  $\Sigma$  be a line arrangement and  $\pi_1(\mathbb{C}^2 - \Sigma) \simeq (\bigoplus_{i=1}^n A_i) \oplus \mathbb{Z}^l$  where  $A_i$  is a free group. Then for any multiple point  $Q$  in the arrangement with  $k$  lines, namely,  $\{l_1, \dots, l_k\}$ , there exists  $r, 1 \leq r \leq n$ , and a projection onto  $A_r$ ,  $\varphi_Q : G \rightarrow G$  such that  $A_r = \langle \varphi_Q(\Gamma_1), \dots, \varphi_Q(\Gamma_k) \rangle \cong \mathbb{F}_{k-1}$ . If  $l_j$  is a line do not pass through the point, then  $\varphi_Q(\Gamma_j) = e$ .*

Moreover, if  $p_1, \dots, p_m$  are the multiple points of  $\Sigma$  and  $n_i$  is the number of lines pass through the point  $p_i$ , then  $G \cong (\bigoplus_{i=1}^m C_i) \oplus B$ , where  $C_i \cong \mathbb{F}_{n_i-1}$ . If  $l$  is a line which does not pass through  $p_i$  and let  $\Gamma$  be its corresponding generator, then  $\text{pr}_i(\Gamma) = e$  (where  $\text{pr}_i$  is the projection onto  $C_i$ ).

## 6. MAIN THEOREMS

**Theorem 6.1.** *Let  $\Sigma \subseteq \mathbb{C}^2$  be a line arrangement which has no pair of parallel lines. Then if*

$$\pi_1(\mathbb{C}^2 - \Sigma) = \bigoplus_{i=1}^r A_i \oplus \mathbb{Z}^l,$$

where  $A_i$  are free groups. Then  $\beta(\Sigma) = 0$ .

*Proof.* Assume by negation that  $\beta(\Sigma) \neq 0$  which means that there is at least one cycle in the graph of  $\Sigma$ . Let us pick a minimal cycle in the following sense. The cycle contains  $r$  points, namely  $\{p_1, \dots, p_r\}$  and  $p_i$  is connected in the cycle only to  $p_{i-1}$  and  $p_{i+1}$  (the indices are taken modulo  $r$ ). In other words, a cycle with no sub-cycles.

By Theorem 5.9, we can write  $G = (\bigoplus_{i=1}^r C_i) \oplus B_1$  where  $B_1$  is not necessarily abelian. If  $l$  is a line that does not pass through the points in the cycle and let  $\Gamma$  be its corresponding generator, then  $\text{pr}_i(\Gamma) = e, 1 \leq i \leq r$ . Define  $N = \langle \Gamma_{X_1}, \dots, \Gamma_{X_t}, \Gamma_1 \cdots \Gamma_n \rangle$ , where  $\Gamma_{X_1}, \dots, \Gamma_{X_t}$  are the generators of lines that do not participate in an intersection point  $p_i, 1 \leq i \leq r$ . Denote  $Z = \Gamma_1 \cdots \Gamma_n$ . Let  $H := G/N$ .

Let  $n_i$  be the number of lines passing through  $p_i$ .

On one hand, since we have  $b := \sum_{i=1}^r (n_i - 1)$  lines participating in the cycle, then if we denote by  $\Gamma_1, \dots, \Gamma_b$  the generators associated with the lines participating in the cycle, then  $H \cong \langle \Gamma_1, \dots, \Gamma_b | \tilde{R}, \tilde{Z} \rangle$  where  $\tilde{R}$  is the relations which are derived from the original relations. Therefore, it is easy to see that  $\text{rank}(\overline{H}) \leq b-1$ .

On the other hand, for  $1 \leq i \leq r$  and  $1 \leq j \leq t$ ,  $\text{pr}_i(\Gamma_{X_j}) = e$  and also  $\Gamma_1 \cdots \Gamma_n \in Z(G) \Rightarrow \text{pr}_i(\Gamma_1 \cdots \Gamma_n) = e \Rightarrow \text{pr}_i(N) = e$  and hence  $C_i/N \cong C_i$ .

$\Rightarrow H = G/N \cong (\bigoplus_{i=1}^r C_i \oplus B_1)/N$ . Since  $C_i \cong \mathbb{F}_{n_i-1}, \overline{C_i} \cong \mathbb{Z}^{n_i-1}$ .

$$\overline{H} = (\bigoplus_{i=1}^r \overline{C_i}) \oplus \overline{B_1}/N$$

$$\Rightarrow \overline{H} = (\bigoplus_{i=1}^r \mathbb{Z}^{n_i-1}) \oplus \overline{B_1}/N \cong \mathbb{Z}^b \oplus \overline{B_1}/N$$

$\Rightarrow \text{rank}(\overline{H}) \geq b$ , a contradiction.  $\square$

We are now ready to prove the converse of Fan's theorem.

**Theorem 6.2.** *Let  $\Sigma \subseteq \mathbb{CP}^2$  be a line arrangement. If*

$$\pi_1(\mathbb{CP}^2 - \Sigma) = \bigoplus_{i=1}^r A_i \oplus \mathbb{Z}^l,$$

where  $A_i$  are free groups. Then  $\beta(\Sigma) = 0$ .

*Proof.* Suppose that  $\Sigma \subset \mathbb{CP}^2$  is an arrangement of complex projective lines such that  $\pi_1(\mathbb{CP}^2 \setminus \Sigma)$  is isomorphic to a direct product of free groups. Let  $L_0 \subset \mathbb{CP}^2$  be a complex projective line which is in general position to  $\Sigma$ . Then  $L_0$  and  $\Sigma$  intersect at double points. Choose an arbitrary complex projective line of  $\Sigma$  and denote this line by  $L_1$ . Note that  $\mathbb{C}^2 \cong \mathbb{CP}^2 \setminus L_0 \cong \mathbb{CP}^2 \setminus L_1$ . By applying the product theorem of Oka and Sakamoto [OkSa], we have

$$\begin{aligned} \pi_1(\mathbb{CP}^2 \setminus (\Sigma \cup L_0)) &= \pi_1((\mathbb{CP}^2 \setminus L_1) \setminus ((\Sigma \setminus L_1) \cup L_0)) \\ &\cong \pi_1((\mathbb{CP}^2 \setminus L_1) \setminus (\Sigma \setminus L_1)) \oplus \pi_1((\mathbb{CP}^2 \setminus L_1) \setminus L_0) \cong \pi_1(\mathbb{CP}^2 \setminus \Sigma) \oplus \mathbb{Z}. \end{aligned}$$

This calculation shows that  $\pi_1(\mathbb{CP}^2 \setminus (\Sigma \cup L_0))$  is a product of free group. Note that

$$\pi_1(\mathbb{CP}^2 \setminus (\Sigma \cup L_0)) = \pi_1((\mathbb{CP}^2 \setminus L_0) \setminus (\Sigma \setminus L_0)).$$

Note that  $\Sigma \setminus L_0$  is a union of complex lines in the complex plane  $\mathbb{CP}^2 \setminus L_0$  such that no two lines of this arrangement are parallel.

By Theorem 6.1  $\beta(\Sigma \setminus L_0) = 0$ , therefore  $\beta(\Sigma) = \beta(\Sigma \setminus L_0) = 0$ .

□

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